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(54) **LN-L SCALE SLIDE RULE**

(57) **Abstract:**

This invention relates to a slide rule having an Ln-L scale arrangement for simplifying the use of a slide rule in computations involving the base e, reducing considerably the number of settings required in comparison to the usual slide rule when this base is used.

The present invention provides a slide rule including the combination of coextensive logarithmic and linear scales in which for 10 divisions of said logarithmic scale there are 2.302585 + divisions of said linear scale, and a cursor movable along said scales and having a hairline against which they may be read enabling the direct finding of powers of e and logarithms to the base e under the hairline of said cursor.

A further object of the invention is to provide an Ln scale which is uniform or linear the same as an L scale, but bears relation to the L scale of the constant log e 10 or $1 \div M = 2.302585 +$ for the full length of the Ln scale compared to 1.0 for the full length of the L scale whereby the Ln scale may be used in problems involving the base e in like manner as the L scale is used with problems involving the base 10.

Another object is to provide an Ln scale which may be combined with other slide rule scales such as C and D and bears relation to the C and D scales of the constant e or Naperian base (2.7182818 +), and may also be combined with the CI and DI scales to find the powers of e, logarithms to the base e, logarithms of proper fractions, powers for negative exponents, and mantissas of logarithms by direct readings cross from the Ln and L scales to the C, D, CI and DI scales.

Another object is to provide a slide rule having Ln, L and C and/or D scales bearing the relationship per scale length graduated in the following ratio:

*

Ln = 2.302585 +

L = 1.0

C = 10.0

D = 10.0 and wherein e of the C and D scales = 2.7182818 + and are coincident with 1 on the Ln scale, the values stated being more accurately identified as --

2.302585092994045684017991454684 + and

2.718281828459045235360287471353 + respectively,

according to the MacMillan Logarithmic and Trigonometric Tables, Page 133.

Still another object is to provide Ln and L scales of linear or uniform progression used in combination with other scales of logarithmic progression involving relative settings of the body and slide of the slide rule whereby multiplication and division with powers, the logarithms of combined operations, the powers of other bases, and hyperbolic functions may be determined on a slide rule of our design.

A further object is to provide a slide rule so scaled as to make possible combined operations with the powers of e and logarithms to the base e, with the results accurate to three or four significant figures and readable directly without the necessity of reading a value off one scale and then setting it on another one in order to continue with another calculation. Thus a series of calculations involving the base e may be performed without the use of a log log scale and without the necessity of reading off intermediate values and resetting scales, the final result only ~~be~~ being directly read on the appropriate scale.

'A'

Still a further object it to provide a slide rule having a scale for the base e capable of doing everything that the L scale does for the base 10.

An additional object is to provide a scale for a slide rule which makes possible the reading of e to any exponent between 0 and 2.3 directly on the C or D scales of the rule, or e to any exponent between 0 and -2.3 directly on the CI or DI scales, as well as e to any power between 1 and 10 on the C or D scales, or e to any power between 1 and -10 on the CI or DI scales.

Another additional object is to provide an Ln scale so related to all logarithmic scales of a slide rule that problems involving any logarithmic scale and the base e are greatly simplified, and particularly so in combined operations, in comparison with such problems when worked on a slide rule having no Ln scale.

With these and other objects in view, our invention consists in the construction, arrangement and combination of the various parts of our Ln-L scale slide rule, whereby the objects above contemplated are attained, as hereinafter more fully set forth, pointed out in our claims and illustrated in detail on the accompanying drawings, wherein:

Figure 1 shows an arrangement of scales for a slide rule wherein our Ln scale is on the slide and arranged back to back with respect to an L scale;

Figure 2 shows another scale arrangement wherein the Ln and L scales are provided on the body of the slide rule;

Figure 3 shows a complete slide rule with the scales of Figure 1 thereon;

Figures 4, 5, 6 and 7 show scale and cursor settings to illustrate the use of the Ln and L scales to find powers of e, logarithms to base e and other values involving the base e; and

Figures 8, 9, 10 and 11 show scale and cursor settings combined with body and slide settings to illustrate how the L and Ln scales are used in combination with logarithmic scales of a slide rule.

On the accompanying drawings we have used in Figures 1, 2 and 3 the reference numerals 10 and 12 to indicate the body portions of a slide rule, and 14 the slide thereof. The body portions 10 and 12 are connected by end members 16 in the usual way, and a cursor 18 of transparent material is supported by slides 20 and 22 that slide along the top and bottom of the rule so that the hairline 24 of the cursor 18 may be matched with various slide rule graduations as required.

The body 10-12 in Figure 1 is provided with such standard scales as Log-Log (LL1, LL2, LL3 and LL4) and the slide and body are provided with the usual C and D scales respectively. The slide 14 is also provided with the usual linear Log scale (L) to which the logarithmic C and D scales are related, and the usual TH and SH scales are also provided on the slide.

The important scale of our invention is the Ln scale which is calibrated in accordance with Napierian or natural logarithms (logarithms to the base e). Napier explained in his book of 1614 the value of e = 2.71828 approximately. Accordingly, 1.0 to our Ln scale is coincident with 2.7182818 + of the C scale in Figure 1 and the D scale in Figure 2, and is designated with a mark identified epsilon (ϵ) throughout the drawings.

The Ln scale is a uniformly divided scale and its length is equal to log e to the base 10, or 2.302585 + . It accordingly bears the relationship of this number to the length 1.0 of the L scale as shown. Thus calibrated, the Ln scale is useful to simplify problems involving the base e when used with the Log scale and/or with any of the other scales that are related (a logarithmic to linear relationship) to the Log scale, as will hereinafter be explained in detail.

Figure 2 shows further scales as follows:

a D scale folded at π (DF)

a C scale folded at π (CF), and

a CI folded scale (CIF) reciprocal to CF, all related to the Log scale so that they may be used in computations involving the base e through their relation with the Log or L scale and its relation to our Ln scale.

Figure 1 also shows DF/M and CF/M scales folded at the value 2.302585 + which may be used in some computations likewise involving the Ln scale as well as the logarithmic scales of our slide rule.

In general, the Ln scale is used for problems with the base e. In explanation of the use of the designation "Ln", this is a symbol now commonly used in mathematical books, being an abbreviation for "natural logarithm". Log 8 is an expression used for "logarithm of 8 to base 10", and likewise Ln 8 is for "logarithm of 8 to base e".

The range of our Ln scale (from 0 to 2.3) is greater than the range of the L scale (from 0 to 1). By computing a "characteristic" one can use the Ln scale to find any power of e; thus, the effective range for powers of e is from 0 to infinity. Also, for many problems it is more convenient to use an Ln scale than Log-Log scales, and in particular it enables one to solve problems with powers of e in combined operations. Since powers of e are read on the C (or D) scale, accuracy to three or four significant figures is obtained no matter how large or how small the numbers are. Our Ln scale thereby saves steps in many computational problems.

Following is a detailed description of the operation of our disclosed slide rule and examples of problems in two parts:

PART 1 (Sections a to g)

(a) Finding logarithms

(b) Powers of e and 10

(c) Finding logarithms of proper fractions

(d) Powers for negative exponents

(e) Finding the characteristic

(f) Extending the range for 10^y and e^y

(g) A shortcut in using Ln 10 = 2.30258

In Part 1 the use of the Ln scale to find powers of e and logarithms to base e is explained. The Ln scale is used with problems in base e just as the L scale is used with problems in base 10. When referring to the scales shown in the various figures of the drawing, the slide and cursor are always set on the rule like the ones pictured in the Figures. In this description we will use the symbol Lnx for the logarithm of x to base e. References to L scales (Ln and L) "on the slide" are the scales of Figure 1 and "on the body" are the scales of Figure 2.

(a) <u>Finding logarithms</u>. Referring to Figure 4, five different settings of the hairline 24 are illustrated together with the values found according to the following:

Rule: To find mantissas of logarithms: Set the number on the C (or D) scale, and read the mantissa for base 10 from L, or for base e from Ln. If L scales are on slide, set on C. If L scales are on body, set on D. Characteristic for base 10 is found by usual method. "Characteristic" for base e is explained in Section (e). Also, see below.

For $1 \leqq x \leqq 10$, we have $0 \leqq \log x \leqq 1$.

For $1 \leqq x \leqq 10$, we have $0 \leqq \text{Ln } x \leqq 2.30258$

For the same domain ($1 \leqq x \leqq 10$), the range of Ln is greater than the range of L.

Examples for practice. Verify that:

1. Ln 1.473 = 0.387	2. Log 1.473 = 0.168
3. Ln 2.34 = 0.850	4. Log 2.34 = 0.369
5. Ln π = 1.145	6. Ln 3.49 = 1.250
7. Ln 4 = 1.386	8. Ln 4.62 = 1.530
9. Ln 5.17 = 1.643	10. Ln 7.35 = 1.995
11. Ln 8.05 = 2.086	12. Ln 9 = 2.197
13. Ln 3.62 = 1.286	14. Log 3.62 = 0.559
15. Ln 1.91 = 0.647	16. Log 1.91 = 0.281
17. Ln 2.66 = 0.978	18. Log 2.66 = 0.425

(b) Powers of e and of 10. In Figure 5 the cursor hairlines 24 are in the same positions as in Figure 4 for Section (a), but the corresponding values for the powers of e and of 10 are given.

Rule: To find powers of e and of 10. Set the exponent of e on Ln, or of 10 on L, and read the power on C (or D). If L scales are on slide, use C; if they are on body, use D. The decimal point of the answer is found by special rules. See Section (f). Also see below.

For $0 \leqq y \leqq 1$, we have $1 \leqq 10^y \leqq 10$.

For $0 \leqq y \leqq 2.30258$, we have $1 \leqq e^y \leqq 10$.

Although the domain of y is greater for base e, the range of 10^y and e^y is the same. The exponents for this range may be set directly on the Ln or L scale.

Examples for practice. Verify that:

1. $\sqrt{e} = e^{0.5}$	$= 1.649$	2. $e^{0.81}$	$= 2.25$
3. $\sqrt{10} = 10^{0.5}$	$= 3.16$	4. $e^{1.15}$	$= 3.16$
5. $e^{1.36}$	$= 3.90$	6. $e^{1.63}$	$= 5.104$
7. $e^{1.875}$	$= 6.52$	8. e^2	$= 7.39$
9. $e^{2.138}$	$= 8.48$	10. $e^{2.252}$	$= 9.51$
11. $10^{0.732}$	$= 5.40$	12. $10^{0.9}$	$= 7.95$
13. $\sqrt{e^3} = e^{1.5}$	$= 4.48$	14. $10^{0.405}$	$= 2.54$
15. $e^{0.1}$	$= 1.105$	16. $e^{0.205}$	$= 1.228$
17. $e^{0.532}$	$= 1.702$	18. $e^{0.01}$	$= 1.0101$
19. $e^{0.99}$	$= 2.691$	20. $e^{1.01}$	$= 2.746$

(c) Finding logarithms of proper fractions. The logarithm of each proper fraction is a negative number. It is written in two ways; for example:

Log 0.5 = -0.301 or 9.699-10; also,

Ln 0.5 = 0.693 or 9.307-10

For slide rule work the form 9.--- -10 is not needed. Figure 6 illustrates cursor settings in relation to the significant scales of the slide rule for problems of this kind and gives the corresponding values.

Rule: To find mantissas of logarithms of numbers between 0.1 and 1, set number on CI (or DI), read mantissa for base 10 on L, for base e on Ln. If L scales are on the slide, use CI. If L scales are on the body, use DI. For smaller numbers, see Section (f). Also, see below.

For $0.1 \leqq x \leqq 1$, we have $-1 \leqq \log x \leqq 0$.

For $0.1 \leqq x \leqq 1$, we have $-2.30258 \leqq \text{Ln}x \leqq 0$.

For the same domain ($0.1 \leqq x \leqq 1$), the range of Ln is greater than the range of L. For x in this domain, the logarithm is read directly from the scale and written with the negative sign. For x not in this domain, the characteristic must be found by a special rule.

Examples for practice. Verify that:

1. Ln 0.15 = -1.897	2. Ln 0.1625 = -1.818
3. Ln 0.19 = -1.661	4. Log 0.19 = -0.721
5. Ln 0.202 = -1.599	6. Log 0.202 = -0.695
7. Ln 0.259 = -1.351	8. Log 0.259 = -0.587
9. Ln 0.3 = -1.204	10. Ln π/10 = -1.158
11. Ln 0.85 = -0.163	12. Ln 0.92 = -0.083
13. Log 0.742 = -0.130	14. Ln 0.742 = -0.298
15. Log 0.363 = -0.440	16. Ln 0.363 = -1.014
17. Log 0.178 = -0.750	18. Ln 0.178 = -1.726
19. Ln 0.120 = -2.120	20. Log 0.12 = -0.921
21. Ln 0.103 = -2.27	22. Log 0.103 = -2.274

(d) Powers for negative exponents. For negative exponents, powers are all less than 1. Hence, they are proper fractions. In Figure 7 the cursor hairlines are shown in the same positions as in Figure 6 for Section (c) but negative exponent values are given.

Rule: To find powers of e and of 10 for negative exponents, set the exponent of e on Ln, or the exponent of 10 on L, and read the power on CI (or DI). If L scales are on slide, use CI; if they are on the body, use DI. The decimal point in the answer is found by special rules. See Section (f). Also, see below.

For $-1 \leqq y \leqq 0$, we have $0.1 \leqq 10^y \leqq 1.0$.

For $-2.30258 \leqq y \leqq 0$, we have $0.1 \leqq e^y \leqq 1.0$.

Although the domain of y is greater for base e, the range of 10^y and of e^y is the same. The exponents for this range may be set directly on the Ln or the L scale.

Examples for practice. Verify that:

1. e^{-2}	= 0.135	2. $10^{-0.8}$	= 0.1585
3. e^{-1}	= 0.368	4. $10^{-0.1}$	= 0.794
5. $e^{-0.2}$	= 0.819	6. $e^{-0.63}$	= 0.533
7. $e^{-0.27}$	= 0.763	8. $e^{-1.54}$	= 0.214
9. $e^{-1.27}$	= 0.281	10. $e^{-2.08}$	= 0.125
11. $e^{-2.29}$	= 0.1013	12. $e^{-1.65}$	= 0.192
13. $10^{-0.67}$	= 0.2138	14. $e^{-0.72}$	= 0.487
15. $10^{-0.44}$	= 0.363	16. $e^{-0.56}$	= 0.571
17. $10^{-0.25}$	= 0.562	18. $e^{-0.99}$	= 0.372
19. $e^{-1.5}$	= 0.223	20. $10^{-0.15}$	= 0.708

(e) Finding the characteristic.

Base 10

The characteristic is the exponent of 10 when the number is expressed in standard form.

Rule: To express a number in standard form: (i) place a decimal point at the right of the first non-zero digit, (ii) start at the right of the first non-zero digit in the original number and count the digits and zeros passed over in reaching the decimal point. The result of the count is the numerical value of the characteristic, or exponent of 10. If the original decimal point is toward the right, the characteristic is positive (+). If the original decimal point is toward the left, the characteristic is negative (-). Indicate that the result of (i) is multiplied by 10 with this exponent.

Examples for practice.

Number	Number in standard form	Characteristic
1. 5,790,000	5.79×10^{6}	6
2. 0.000283	2.83×10^{-4}	-4
3. 44	4.4×10^{1}	1
4. 0.623	6.23×10^{-1}	-1
5. 8.15	8.15×10^{0}	0
6. 461,328	4.61328×10^{5}	5
7. 0.0000005371	5.371×10^{-7}	-7
8. 0.0306	3.06×10^{-2}	-2

Base e

The term "characteristic" as used here will mean the number to which a reading from the Ln scale must be added to account for logarithms not in its range.

Rule: First express the number in standard form. Read the logarithm of the first factor directly from Ln, as in Section (a). Multiply 2.30258 by the exponent of 10 in the second factor. If the exponent is positive, add this result to the direct reading. If the exponent is negative, subtract the result from the direct reading.

Examples for practice. Verify that the logarithm to base e for the examples at the left is as follows:

1. $\text{Ln } 5.79 \times 10^{6} = 1.756 + 6(2.303) = 15.574$
2. $\text{Ln } 2.83 \times 10^{-4} = 1.040 - 4(2.303) = -8.170$
3. $\text{Ln } 4.4 \times 10^{1} = 1.482 + 2.303 = 3.785$
4. $\text{Ln } 6.23 \times 10^{-1} = 1.829 - 2.303 = -0.474$
5. $\text{Ln } 8.15 \times 10^{0} = 2.098 - 0 = 2.098$
6. $\text{Ln } 4.61 \times 10^{5} = 1.528 + 5(2.303) = 13.041$
7. $\text{Ln } 5.371 \times 10^{-7} = 1.681 - 7(2.303) = -14.437$
8. $\text{Ln } 3.06 \times 10^{-2} = 1.118 - 2(2.303) = -3.487$

For $0 < x < \infty$, we have $-\infty < \log x < +\infty$.

For $0 < x < \infty$, we have $-\infty < \text{Ln } x < +\infty$.

(f) Extending the range for 10^{y} and e^{y}.

Base 10

For y not in the interval between 0 and 1, the standard method of finding 10^{y} first expresses it as the product of two factors. Thus $10^{2.5} = 10^{2} \times 10^{0.5}$. One factor has an integral exponent. The other factor has a fractional exponent in the interval 0 to 1. The second factor is computed by the methods of Section (b), and Section (d). The first factor then determines the position of the decimal point in the final answer.

Examples for study.

1. $10^{2.5} = 10^{2} \times 10^{0.5} = 10^{2} \times 3.16 = 316.$
2. $10^{4.26} = 10^{4} \times 10^{0.26} = 10^{4} \times 1.82 = 18{,}200.$
3. $10^{-5.38} = 10^{-5} \times 10^{-0.38} = 10^{-5} \times 0.417 = 0.00000417.$
4. $10^{-2.71} = 10^{-2} \times 10^{-0.71} = 10^{-2} \times 0.195 = 0.00195.$

For $-\infty < y < +\infty$, we have $0 < 10^y < +\infty$. The effective range for powers of 10 is infinite. Three or four significant figures of e^y can be found.

Examples for practice. Verify that:

1. $10^{3.916} = 8,240$ 2. $10^{-3.916} = 0.0001214$
3. $10^{5.023} = 105,400$ 4. $10^{-5.023} = 0.000,009,48$
5. $10^{14.622} = 4.19 \times 10 = 419,000,000,000,000$
6. $10^{-23.877} = 0.000,000,000,000,000,000,000,001,327$

Base e

For y not in the interval $0 \leqq y \leqq 2.30258$, express e^y as the product of two factors. For example, $e^{3.5} = e^{2.3} \times e^{1.2} = 10e^{1.2}$. One method of finding these factors is to divide the exponent y by 2.30258 (or a rounded value of this divisor, such as 2.303), to determine an integral quotient, q, and a remainder, r. Then $y = 2.303q+r$, and $e^y = e^{2.303q+r} = e^{2.303q} \times e^r = (e^{2.303})^q \times e^r = 10^q \times e^r$.

The value of the second factor is computed by the methods of Section (b), and Section (d). The first factor is used to determine the decimal point.

Examples for study.

1. To find $e^{6.54}$, first divide 6.54 by 2.303, obtaining quotient 2 and remainder 1.934. Then $e^{6.54} = 10^2 \times e^{1.934}$. Set cursor to 1.934 on Ln. Read 6.92 on C (or D). Then answer is $100 \times 6.92 = 692$.
2. Find $e^{-6.54}$. As in Example 1, $e^{-6.54} = 10^{-2} \times e^{-1.934}$. Set cursor hairline to 1.934 on Ln. Read 0.114 on CI (or DI). Then $e^{-6.54} = 0.00144$.

3. Find $e^{17.4}$. Divide 17.4 by 2.303, obtaining quotient ("characteristic") 7 and remainder 1.28. Set hairline to 1.28 of Ln. Read 360 on C (or D). Multiply by 10^7.

For $-\infty < y < +\infty$, we have $0 < e^y < +\infty$. The effective range for powers of e is infinite. Three or four significant figures of e^y can be found.

(g) <u>A short cut in using Ln10 = 2.30258.</u>

To extend the range of Ln the number 2.30258 is needed. Suppose that, to save work, the number 2.3 is used. Some error will of course occur. For example, the remainder in division will be too large. How can we easily correct for this error? The following simple rule will serve:

Rule: Take 1 per cent of the quotient and divide it by 4. Subtract the result from the remainder to obtain the correct remainder to set on Ln.

Example. Find $e^{17.4}$ (Compare with Example 3, under Base e). Divide:

```
       7.                       7.
2.3/17.4     or    2.30258/17.40000
    16.1                   16.11806
     1.3                    1.28194
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Take 1% of 7; 0.01 x 7 = 0.07; Divide by 4. 0.07 ÷ 4 = 0.02, approx. Subtract 0.02 from 1.3, to obtain 1.28, the corrected remainder. Then $e^{17.4} = e^{1.28} \times 10^7$. The basis of this rule is explained below.

Consider $x = e^n$. Divide n by 2.30258, and denote the integral part of the quotient by q and the remainder by r.

Then, $n = 2.30258q + r, \quad r < 2.30258$.

We now propose to use 2.3 as divisor in place of 2.30258. We require the quotient to again be q, but get a new remainder which we denote by R, where $R > r$. Then,

$n = 2.3q+R$, where $R < 2.3$.

Subtracting this from the former equation, we have $0 = 0.00258q+r - R$, or $r = R - 0.00258q$.
Thus the error, $R - r$, in the remainder is 0.00258q. If this is rounded off to 0.0025q, it expresses one-fourth of 1 per cent of the quotient.
When the slide rule is used to divide by 2.30, proceed as follows: Set 2.30 of C over 17.4 of D. Under 1 of C read 7.56 on D. The integral part, or "characteristic", is 7. Multiply the decimal fraction 0.56 by 2.3, using the C and D scales. Obtain 1.29 as the reduced exponent of e.

Figure 1, the quotient may be obtained by merely setting the exponent on DF/M and reading the quotient on D. The relation between readings on the D and the DF/M scales may be indicated symbolically as follows:

(D) x 2.30 = (DF/M) and (DF/M) ÷ 2.30 = (D).

For some purposes and for some exponents, this slide rule method is not sufficiently accurate.

Examples for practice.

1. Find $e^{7.61}$. Divide 7.61 by 2.3; quotient 3, remainder 0.71. Correction is 0.03/4 = .01. Hence $e^{7.61} = e^{0.70} \times 10^3$, or 2,018.

2. Find $e^{-6.95}$. Divide 6.95 by 2.3 to get characteristic 3 and remainder 0.05 . Correction is 0.03/4 = 0.0075 . Corrected remainder is 0.0425 . Hence $e^{6.95} = e^{-0.0425} \times 10^{-3}$. Set 0.0425 on Ln, read 0.958 on CI. Point off 3 places to the left, to get 0.000958 .

3. Find e^{9} . Divide 9 by 2.3 . Characteristic is 3, remainder 2.1. Correction is 0.03/4 = 0.0075, or .01 . $e^{9} = e^{2.09} \times 10^{3}$. Set 2.09 on Ln, read 8.1 on C. Then e^{9} = 8100.

4. Find e^{-9}, or $e^{-2.09} \times 10^{-3}$. Set 2.09 on Ln. Read on 0.125 on CI, point off 3 places to left to find e^{-9} = 0.000123 .

PART 2 (Sections h to n)

(h) Multiplication with powers

(i) Division with powers

(j) Examples for practice

(k) Logarithms of combined operations

(l) Powers of other bases

(m) Hyperbolic functions

(n) Applied problems

In Part 2 the use of the L and Ln scales in combination with other scales will be explained. The methods used when the L scales are on the body (Figure 2) differ from those used when they are on the slide (Figure 1). Instructions for both types of slide rule are given. In solving problems, first express the numbers in standard form as explained in Section (f) and Section (g). The calculations are carried through within the ranges of L and Ln provided, and the decimal points are determined by special rules.

(h) Multiplication with powers. The scales are set as in Figure 8 to find $16.8 \times e^{1.15}$ with the Ln and L scales on the slide of the rule.

Notice that when 1 of the C scale is set over 16.8 of the D scale, the product of 16.8 and any number set on C is read on D. But by setting the cursor hairline over 1.15 of Ln the value of $e^{1.15}$ is automatically set on C. This number (actually 3.16) does not have to be read. The product is on D. With the log log scales, this value (3.16) must be read and transferred to C before the multiplication can be started.

Rule for $a \cdot e^y$. If L scales are on slide, set 1 of C over a on D. Move cursor hairline to y of Ln. Read figures of answer on D. Determine the decimal point by standard form method. If L scales are on body, begin with e^y. Set cursor hairline to y on L. Set 1 of C under cursor hairline. Move cursor to a of C. Read answer on D. With powers of 10, use L in the same way.

When the L scales are on the body of the rule, the settings are made as in Figure 9.

(i) Division with powers. Remember that division is the opposite of multiplication. The scales pictured in Figures 8 and 9 for Section (h) are set to divide 530 by $e^{1.15}$, that is, to find $530/e^{1.15}$ using D and C, or $530/e^{-1.15}$ using D and CI.

L scales on slide (Figure 1).

Rule: To divide a/e^y, set y on Ln over a on D. Under 1 of C read a/e^y on D.

(j) Examples for practice.

1. Find 2.79 $e^{1.945}/3.82$. Set hairline over 2.79 on D. Move slide so 3.82 on C is under hairline. Move hairline to 1.945 on Ln. Read 5.12 on D.
2. Find 17.35 $e^{1.226}$ sin 43° . Set 1 of C over 17.35 on D. Move hairline over 1.226 on Ln. Move right index of C under hairline. Move hairline to 43 on S. Read 40.3 on D.
3. Find $0.0000452e^{7.61}$ (see Ex. 1, p. 12). Write the work in standard form;
$4.52 \times 10^{-5} \times e^{0.70} \times 10^{3} = 4.52 \times e^{0.70} \times 10^{-2}$.
Set index of C over 452 on D. Move hairline over 0.70 on Ln. Read 912 on D. Answer is 0.0912.
4. Find $5.27^{2}e^{12.7}$. First rewrite $e^{12.7}$ as $e^{1.19} \times 10^{5}$. Set hairline over 5.27 on √. Turn rule over, and set index of C under hairline. Move hairline to 1.19 on Ln. Read 910 on D. Note 5.27^{2} is about 30 or, roughly, 3 x 10. Also $e^{1.19}$ is about 3. Answer, then, is about $3 \times 3 \times 10 \times 10^{5}$ or 9×10^{6} . Correct to three significant figures, answer is 9.10×10^{6} .

L scales on body (Figure 2).

Rule: To divide a/e^{y}, set 1 of C under y of Ln. Move hairline over a on D. Read a/e^{y} on C under the hairline.

1'. Find 2.79 $e^{1.945}/3.82$. Set hairline over 1.945 of Ln. Set slide so 3.82 of C is under hairline. Move hairline over 2.79 on C. Read 5.12 on D.

2'. Find 17.35 $e^{1.226}$ sin 43°. Set hairline over 1.226 on Ln. Move slide so 17.35 on CI is under hairline. Move hairline to 43 on S. Read 40.3 on D.

3'. Find $0.0000452e^{7.61}$ (see Ex. 1, p. 12). Write the work in standard form:

$4.52 \times 10^{-5} \times e^{0.70} \times 10^{3} = 4.52 \times e^{0.70} \times 10^{-2}$.

Set hairline over 0.70 on Ln. Move slide so index of C is under hairline. Move hairline over 452 on C. Read 912 on D. Answer is 0.0912.

4'. Find $5.27^2 e^{12.7}$. First rewrite $e^{12.7}$ as $e^{1.19} \times 10^5$. Set hairline over 1.19 on Ln. It is not convenient to use the A scale for 5.27^2 . Move slide so 527 on CI is under hairline. Move hairline to 527 on C. Read 910 on D. Note 5.27^2 is about 30, or, roughly, 3×10. Also $e^{1.19}$ is about 3. Answer, then, is about $3 \times 3 \times 10 \times 10^5$ or 9×10^6 . Correct to three significant figures, answer is 9.10×10^6 .

(k) Logarithms of combined operations. The scales of Figure 10 are set to find ln 6.78/3.24 and log 6.78/3.24 with L scales on the body.

First, divide 6.78 by 3.24 in the usual way using C and D scales. Move cursor over 1 of C. Read ln 6.78/3.24 on Ln and read log 6.78/3.24 on L. If L scales are on the slide, close rule (move slide so C and D indexes coincide) before reading from Ln or L. Or, if you prefer, first set slide so 1 of C is over 3.24 on D. Move hairline over 6.78 on D. Read logarithm from Ln or L.

In Figure 11 comparable settings are shown when the L scales are on the slide.

Rule: To find ln a/b or log a/b, set b on C over a on D. At index of C read ln on Ln and log on

L. Characteristics must be found by a special rule. If L scales are on slide, set 1 of C over b on D. Move indicator over a on D. Read logarithm from Ln on L. This method requires only one setting of the slide.

(l) Powers of other bases. Sometimes powers of bases other than 10 or e are needed. If a slide rule has Log Log scales, they may be used to find these powers within the range provided. If the slide rule does not have Log Log scales, or if the power is outside the scale range provided, one of the following methods may be used.

Using base e

Examples

1. Find $1.5^{2.4}$. Write $e^x = 1.5$. Set 1.5 on C (or D), find x = 0.405 on Ln. Then $(1.5)^{2.4} = (e^{0.405})^{2.4} = e^{0.972}$ by multiplying the exponents. Using Ln again, set 0.972 on Ln, read the answer 2.65 on C (or D). This solution can be expressed in logarithmic form as follows: $\ln 1.5^{2.4} = 2.4 \ln 1.5$.

2. Find $18.5^{-6.37}$. Set $y = 18.5^{-6.37}$. Then $\ln y = -6.37 \times \ln 18.5$. Now $\ln 18.5 = \ln 1.85 \times 10 = 0.615 + 2.303$ or 2.918. Now $-6.37 \times 2.918 = 18.58$ and $18.58 \div 2.3 = 8$ with remainder of 0.18. But 1% of 8 = .08, and $.08 \div 4 = .02$, so with the correction $0.18 - 0.02 = 0.16$, we have to find $e^{-0.16} \times 10^{-8}$. Set hairline over 0.16 on Ln, read 0.852 on CI (or DI). The quotient 8 tells us the answer is $0.852 \times 10^{-8} = 8.52 \times 10^{-9}$.

3. Find $0.88^{0.25}$. Set $y = 0.88^{0.25}$. Then $\ln y = 0.25 \times \ln 0.88$. Write $\ln 0.88 = \ln 8.8 \times 10^{-1}$. Set hairline over 8.8 of C (or D), read 2.175 on Ln. Then $\ln 8.8 \times 10^{-1} = 2.175 - 2.303 = -0.128$, and $0.25 \times (-0.128) = -0.032$. Set hairline over 0.032 on Ln, read answer 0.968 on CI (or DI).

Using base 10

Examples

1. Find $1.5^{2.4}$. Write $10^{x} = 1.5$. Set 1.5 on C (or D), find $x = 0.176$ on L. Then $(1.5)^{2.4} = (10^{0.176})^{2.4} = 10^{0.422}$, by multiplying exponents. Set 0.422 on L, read the answer 2.65 on C (or D). This solution can be expressed in logarithmic form as follows: $\log 1.5^{2.4} = 2.4 \log 1.5$.

2. Find $18.5^{-6.37}$. Set $y = 18.5^{-6.37}$. Then $\log y = -6.37 \times \log 18.5$. Now $\log 18.5 = 0.267 + 1 = 1.267$. Now $-6.37 \times 1.267 = -8.07$. We must now set the hairline over 0.07 on L, reading from right to left, or subtract 8.07 from 10.00 - 10 to write the logarithm with a positive mantissa, namely 1.93 - 10. Set the hairline over 0.93 on L and read 8.50 on C (or D). Then the results is 8.50×10^{-9} .

3. Find $0.88^{0.25}$. Set $y = 0.88^{0.25}$. Then $\log y = 0.25 \times \log 0.88$. Write $\log 0.88 = \log 8.8 \times 10^{-1}$. Set hairline over 8.8 on C (or D), read 0.944 on L. Then $\log 88 \times 10^{-1} = 0.944 - 1$, or $3.944 - 4$. $0.25\ (3.944 - 4) = 0.986 - 1$. Set hairline over 0.986 of L, read 9.68 on C (or D). Then answer is 0.968 .

(m) Hyperbolic functions. The Ln scale is very helpful in finding values of the hyperbolic functions. This is especially true for a rule having the scales shown in Figure 2 which does not provide Log-Log scales or hyperbolic function scales. However, even with the type of scales shown in Figure 1 on which these extra scales are available, the Ln scale simplifies the work in problems that fall outside the range of the scales provided.

By definition, $\sinh x = (e^x - e^{-x})/2$, or $\sinh x = (e^x/2) - (e^{-x}/2)$.

By definition, $\cosh x = (e^x + e^{-x})/2$, or $\cosh x = (e^x/2) + (e^{-x}/2)$.

By definition, $\tanh x = (e^x - e^{-x})/(e^x + e^{-x}) = (\sinh x)/(\cosh x)$.

Rule: To find sinh x or cosh x, set hairline over x on Ln, read e^x on C (or D) and e^{-x} on CI (or DI). For sinh x, subtract e^{-x} from e^x and divide the result by 2. For cosh x, add e^x and e^{-x}, and divide by 2. To find tanh x, use Ln to find e^x and e^{-x}; divide their difference by their sum.

Rule: For $x > 3$, $\sinh x = \cosh x = e^x/2$ can be found by setting the index of the C scale over 5 on the D scale, moving the hairline to x on Ln, and reading the result on D.

Examples.

1. Find sinh 5.4 or $e^{5.4}/2$. Divide 5.4 by 2.303, obtaining quotient 2 and remainder 0.794. $(e^{5.4}/2) = (e^{0.794}/2)10^2$. Set right hand index of C over 5 of D. Move hairline to 0.794 on Ln. Read 1.107 on D. This must be multiplied by 10^2, so sinh 5.4 = 110.7 .

2. Find sinh 24 = $e^{24}/2$. First write e^{24} = $e^{0.975} \times 10^{10}$. Using Ln, find $e^{0.975}$ = 2.65. Then 2.65/2 = 1.32, so sinh 24 = 1.32×10^{10} .

Extending the ranges. Remember that if $x > 2.3$, you must first divide x by 2.3 and correct the remainder. The corrected remainder is set on Ln instead of x, and the integral quotient q is the exponent of 10 such that the factor 10^q determines the position of the decimal point.

For $x > 3$, we have $(e^{-x}/2) < 0.025$. Hence, for $x > 3$, sinh x = cosh x = $e^x/2$, approximately, and tanh x = 1. For $x < 0.10$, we have sinh x = x, cosh x = 1, and tanh x = x, approximately.

Figure 1, the values of e^x and e^{-x} for $x < 23$ can be found directly on the Log Log scales by using the DF/M scale. For $x > 10$ the accuracy is poor. For $x > 23$, the Log Log scales are useless.

Examples.

3. Find cosh 4.8 = $e^{4.8}/2$. Write $e^{4.8} = e^{0.195} \times 10^2$. Set left index of C over 5 on D. Move hairline over 0.195 on Ln. Read $e^{0.195}/2$ = 0.608 on D. Then cosh 4.8 = 60.8 .

4. Find tanh 1.3. Using Ln, read $e^{1.3}$ = 3.67 on C (or D) and $e^{-1.3}$ = 0.273 on CI (or DI). Then 3.67 - 0.273 = 3.397 and 3.67 + 0.273 = 3.94; hence tanh 1.3 = 0.862 .

(n) Applied problems.

1. As an extraordinary example consider the following quotation:

"The total N for the entire line is N = 0.1118 x 2000 = 223.6 nepers, and the ratio of input to output current

$$"I_s/I_r = e^{223.6} = 10^{97}\ "*$$

Calculate $e^{223.6}$.

By the method of Section (g), above, we divide 223.6 by 2.3, and correct the remainder.

```
      97
23/223.60          0.01 x 97 = 0.97
   207.
    16.6           1/4 x .97 = 0.24
    16.1
      .50          0.50 - .24 = 0.26
```

Set hairline over 0.26 of Ln.

Read 1.297 on C.

The result is 1.297×10^{97}. The result found by logarithms is 1.286×10^{97}. The error is 0.85%, or under 1%, and occurs because the correction formula uses 0.0025 instead of 0.00258.. This shows that the method using Ln is sufficiently accurate for all exponents up to 100; such large exponents are exceedingly rare.

2. A table of standard sizes for rectangular wire may be made by inserting 38 geometric means between the diameter (0.46 inch) of Gauge 0000 and the diameter (0.005 inch) of Gauge 36 of the American Wire Gauge.

Calculate the common ratio $r = \sqrt[39]{\frac{0.4600}{0.005}}$,

and compute the 36th term.

*Ware, Lawrence A. and Reed, Henry R. <u>Communication Circuits.</u> (New York, 1942), John Wiley and Sons, Inc. p. 49.

First note that $r = (460/5)^{1/39}$, or $(92)^{1/39}$. Write $10^x = 92$. Set 92 on C (or D), find mantissa of x, or 0.964 on L. Then x = 1.964. Then $r = (10)^{1.964/39} = 10^{0.0503}$; set 0.0503 on L, read r = 1.123 on C (or D). The 36th term is 0.005×1.123^{35}, or $0.005 \times 10^{35} \times (0.0503) = 0.005 \times 10^{1.761}$. Set hairline over 0.761 on L, read 5.77 on C (or D). Finally, compute 0.005 x 10 x 5.77 = 0.289 inch, or 289 mills, approximately.

3. The formula for the current in a certain circuit

   is $i = 1.25\ (1 - e^{-80t}),\ 0 \leqq t \leqq 0.01.$

   Find i for t = 0.006; that is,

   $i = 1.25\ (1 - e^{-80 \times 0.006}) = 1.25\ (1 - e^{-0.48}).$

   Set hairline over 0.48 on Ln, read $e^{-0.48} = 0.619$ on CI (or DI). Then i = 1.25 (1 - 0.619) = 1.25 x 0.381 = 0.476.

4. In a problem similar to 3, above, the formula is

   $i = 4\ (1 - e^{-40t}),\ 0 \leqq t \leqq 0.02$. Find i for t = 0.015; that is $i = 4\ (1 - e^{-0.60})$.

   Answer: 1.804.

Having described our invention and the operation of a number of the disclosed scales in combination with each other, it is believed obvious how our Ln-L scale slide rule simplifies problems to the base e, and is operable in combination with many different types of logarithmic scales of a slide rule such as the CF, DF, CF/M, DF/M, CIF, LL, TH and SH scales shown in Figures 1, 2 and 3. These scales are illustrated in their proper relations to the Ln and L scales. Examples are disclosed of logarithmic scales with which the Ln scale can be used, and any other logarithmic scales usable with the L scale can also be used in an obvious manner with the Ln scale for values to the base e instead of the base 10.

Some changes may be made in the construction and arrangement of the parts of our slide rule without departing from the real spirit and purpose of our invention, and it is our intention to cover by our claims any modified forms of structure or use of mechanical equivalents which may reasonably be included within their scope.

The embodiments of the invention in which an exclusive property or privilege is claimed are defined as follows:

1. In a slide rule, the combination of an L scale of unit length, an Ln scale coextensive therewith and graduated in equal increments totalling 2.302585 +, and at least one logarithmic scale having graduations from 1 to 10 coextensive with said Ln and L scales.

2. A slide rule of the character disclosed comprising the combination of an L scale of unit length, an Ln scale coextensive therewith and graduated in equal increments totalling 2.302585 +, and a logarithmic scale having graduations from 1 to 10 coextensive with said Ln and L scales, whereby powers of e and logarithms to the base e may be directly determined from said Ln scale and said logarithmic scale, when used in combination.

3. In a slide rule, the combination of an L scale of unit length, an Ln scale coextensive therewith and graduated in equal increments totalling 2.302585 +, and at least one logarithmic scale having graduations from 1 to 10 coextensive with said Ln and L scales, whereby logarithms for base e may be read directly on said Ln scale when a number is set on said logarithmic scale, and powers of e can be read on said logarithmic scale when the number is set on said Ln scale.

4. In a slide rule, the combination of an Ln scale graduated in increments totalling 2.302585 + and a logarithmic scale coextensive therewith having graduations from 1 to 10 whereby powers of e and logarithms to the base e may be determined from said Ln scale in conjunction with said logarithmic scale.

5. In a slide rule, the combination of an Ln scale graduated in equal increments totalling 2.302585 + and an inverted logarithmic scale coextensive with said Ln scale having graduations from 10 to 1 whereby powers of e for negative exponents, and logarithms of proper fractions to the base e, may be determined from said Ln scale and said inverted logarithmic scale when used in conjunction with each other.

6. A slide rule comprising in combination, coextensive logarithmic and linear scales in which for 10 main increments of said logarithmic scale there are 2.302585 + main increments of said linear scale, a log scale coextensive with said linear and logarithmic scales and of unit length, and a cursor movable along said scales and having a hairline against which they may be read, enabling the finding of powers of e and logarithms to the base e under the hairline of said cursor in combined slide rule operations.

7. In a slide rule, the combination of coextensive logarithmic and linear scales in which for 10 increments of said logarithmic scale there are 2.302585 + increments of said linear scale, and a cursor movable along said scales and having a hairline against which they may be read enabling the direct finding of powers of e and logarithms to the base e under the hairline of said cursor.

8. A slide rule comprising the combination of coextensive logarithmic, inverted logarithmic and linear scales in which for 10 increments of the logarithmic scales there are 2.302585 + increments of said linear scale, and a cursor movable along said scales and having a hairline against which they may be read enabling the finding of powers of e both positive and negative and logarithms to the base e for both whole numbers and proper fractions under the hairline of said cursor.

9. In a slide rule, the combination of a plurality of coextensive logarithmic scales and a linear scale in which for 10 increments of said logarithmic scales there are 2.302585 + increments of said linear scale enabling the finding of powers of e and logarithms to the base e, one of said logarithmic scales being inverted for finding logarithms of proper fractions and the powers for negative exponents.

10. In a slide rule, the combination of a body and slide having multiple logarithmic scales and a linear scale coextensive with said logarithmic scales in which for 10 increments of said logarithmic scales there are 2.302585 + equal increments of said linear scale, a cursor movable along said scales and having a hairline against which the values on said scales may be read enabling the finding of powers of e and logarithms to the base e under said hairline, some of said logarithmic scales being inverted for finding the logarithms of proper fractions and the powers for negative exponents, for multiplication and division with powers, for logarithms of combined operations and for values of hyperbolic functions.

11. In a slide rule, the combination of a body and slide having multiple logarithmic scales and a linear scale coextensive with said logarithmic scales in which for 10 increments of said logarithmic scales there are 2.302585 + equal increments of said linear scale, a cursor movable along said scales and having a hairline against which the values on said scales may be read enabling the finding of powers of e and logarithms to the base e under said hairline, some of said logarithmic scales being inverted and others being folded at 2.302585 + for finding the logarithms of proper fractions, the powers for negative exponents, for multiplication and division with powers, logarithms of combined operations and values of hyperbolic functions.

12. In a slide rule, the combination of a body and slide having multiple logarithmic scales and a linear scale coextensive with said logarithmic scales in which for 10 increments of said logarithmic scales there are 2.302585 + equal increments of said linear scale, a cursor movable along said scales and having a hairline against which the values on said scales may be read enabling the finding of powers of e and logarithms to the base e under said hairline, some of said logarithmic scales being inverted and others being log log scales for finding the logarithms of proper fractions, the powers for negative exponents, for multiplication and division with powers, logarithms of combined operations and values of hyperbolic functions.

13. In a slide rule, the combination of a body and slide having multiple logarithmic scales and a linear scale coextensive with said logarithmic scales in which for 10 increments of said logarithmic scales there are 2.302585 + equal increments of said linear scale and a log scale coextensive with said linear scale and said logarithmic scales and of unit length, a cursor movable along said scales and having a hairline against which they may be read enabling the finding of powers of e and logarithms to the base e under the hairline of said cursor, some of said logarithmic scales being inverted, others being folded, and at least one being both inverted and folded for finding the logarithms of proper fractions, the powers for negative exponents, for multiplication and division with powers, logarithms of combined operations and values of hyperbolic functions.

14. In a slide rule, the combination of a body and slide having multiple logarithmic scales and a linear scale coextensive with said logarithmic scales in which for 10 increments of said logarithmic scales there are 2.302585 + equal increments of said linear scale and a log scale coextensive with said linear scale and said logarithmic scales and of unit length, a cursor movable along said scales and having a hairline against which they may be read enabling the finding of powers of e and logarithms to the base e under the hairline of said cursor, some of said logarithmic scales being inverted, others being folded, and still others being log log scales for finding the logarithms of proper fractions, the powers for negative exponents, for multiplication and division with powers, logarithms of combined operations and values of hyperbolic functions.

*

FIG. 1

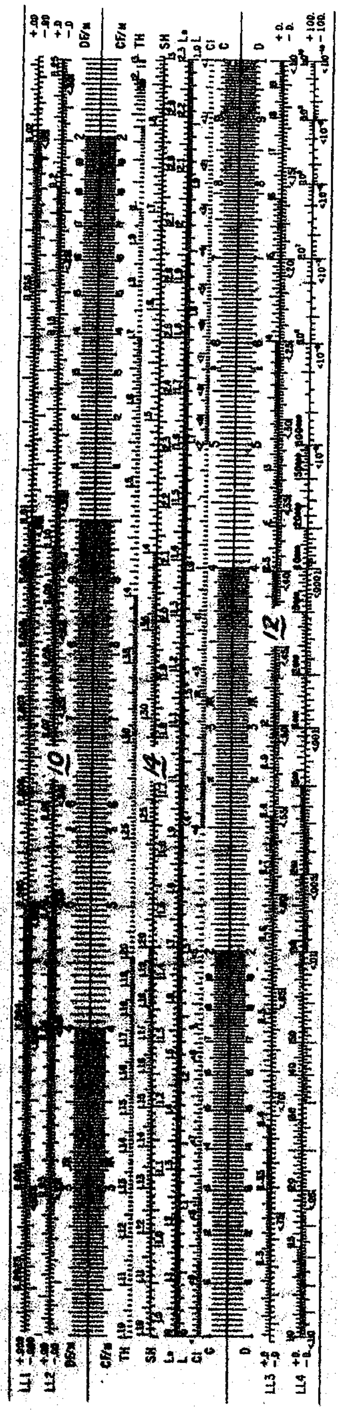

FIG. 2

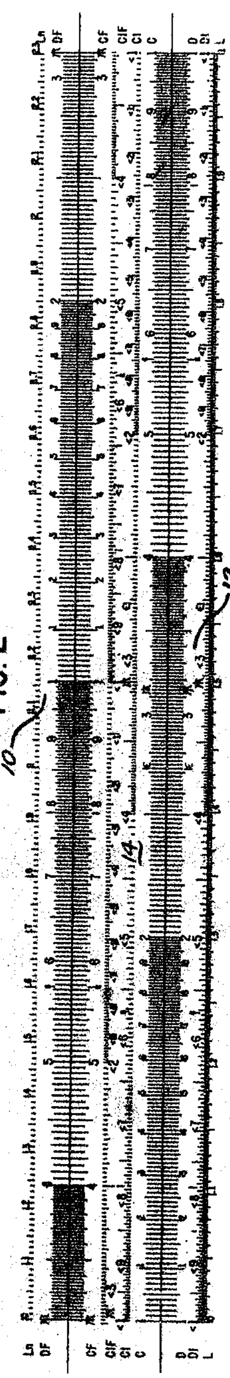

FIG. 3

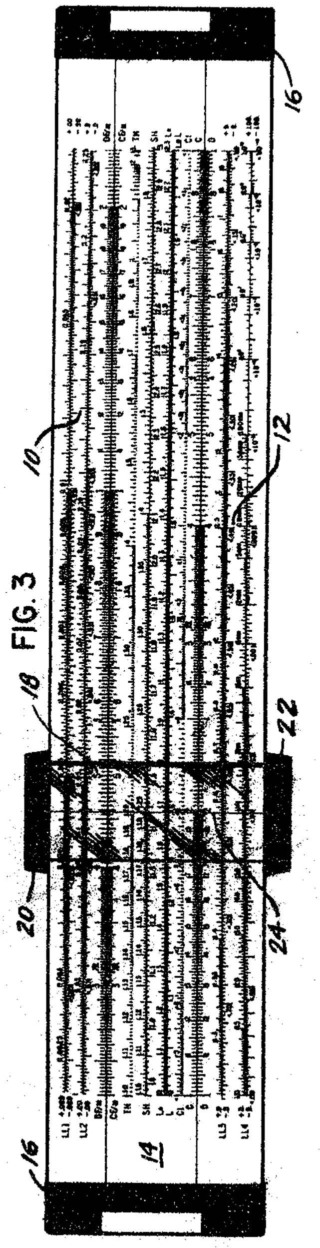

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FIG. 4

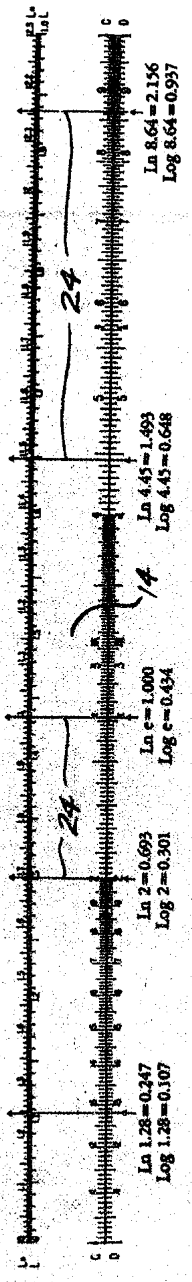


FIG. 5

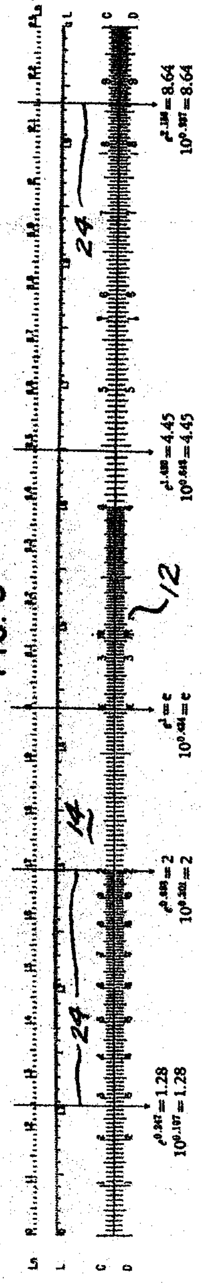


FIG. 6

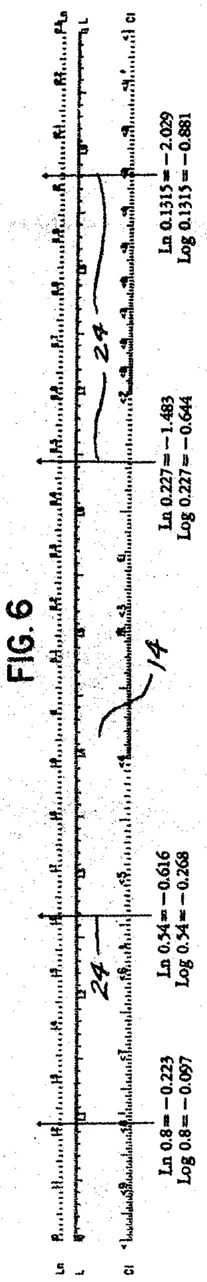
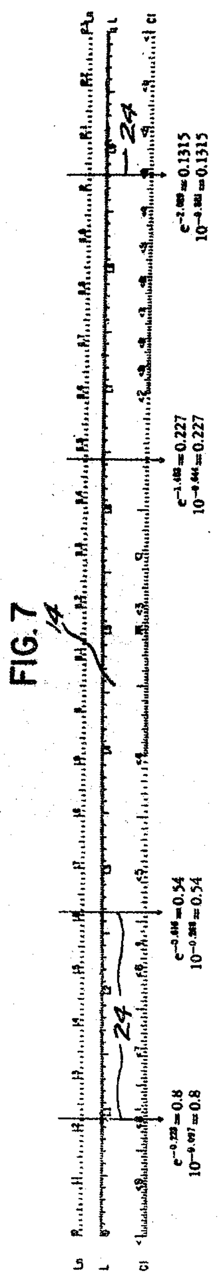


FIG. 7



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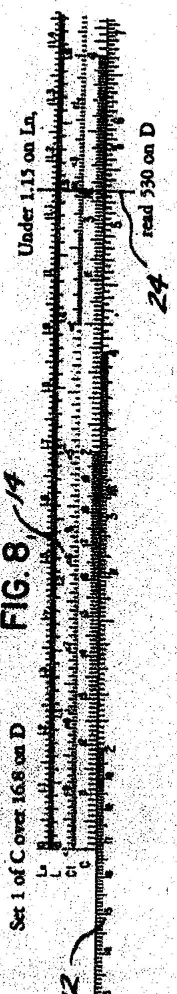

FIG. 8

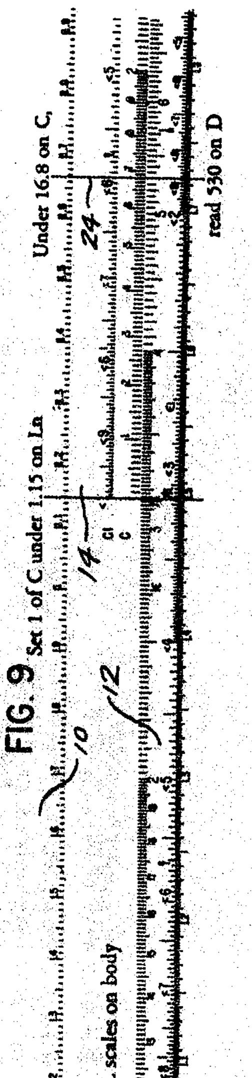

FIG. 9

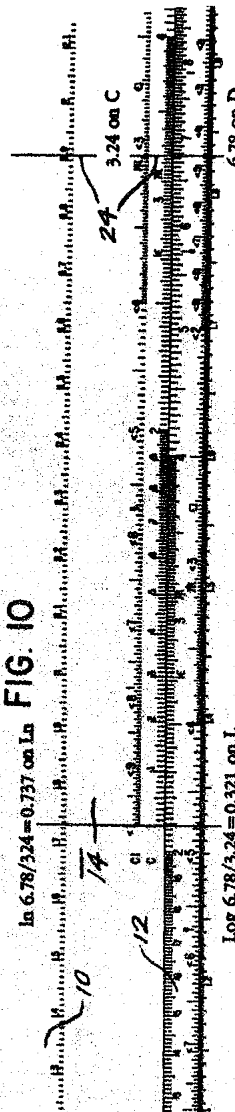

FIG. 10

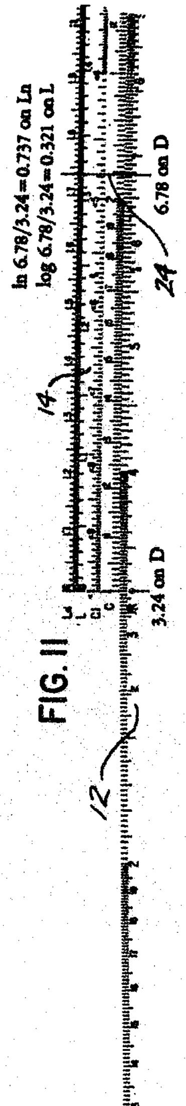

FIG. 11

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